

# On Separation of Minimal Riesz Energy Points on Spheres in Euclidean Spaces

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## Abstract

Let  $S^d$  denote the unit sphere in the Euclidean space  $\mathbb{R}^{d+1}$  ( $d \geq 1$ ). Let  $N$  be a natural number ( $N \geq 2$ ), and let  $\omega_N := \{x_1, \dots, x_N\}$  be a collection of  $N$  distinct points on  $S^d$  on which the minimal Riesz  $s$ -energy is attained. In this paper, we show that the points  $x_1, \dots, x_N$  are well-separated for the cases  $d-1 \leq s < d$ .

## 1 Introduction.

Let  $S^d$  denote the unit sphere in the Euclidean space  $\mathbb{R}^{d+1}$  ( $d \geq 1$ ). Let  $N$  be a natural number ( $N \geq 2$ ), and let  $\omega_N := \{x_1, \dots, x_N\}$  be a collection of  $N$  distinct points on  $S^d$ . The Riesz  $s$ -energy ( $s \geq 0$ ) associated with  $\omega_N$ ,  $E_s(\omega_N)$ , is defined by

$$E_s(\omega_N) := \begin{cases} \sum_{i \neq j} \frac{1}{|x_i - x_j|^s}, & \text{if } s > 0, \\ \sum_{i \neq j} \log \frac{1}{|x_i - x_j|}, & \text{if } s = 0. \end{cases}$$

Here  $|\cdot|$  denotes the Euclidean norm. We use  $\mathcal{E}_s(S^d, N)$  to denote the  **$N$ -point minimal  $s$ -energy** over  $S^d$  defined by

$$\mathcal{E}_s(S^d, N) := \min_{\omega_N \subset S^d} E_s(\omega_N), \quad (1.1)$$

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where the infimum is taken over all  $N$ -point subsets of  $S^d$ . If  $\omega_N \subset S^d$  is such that

$$E_s(\omega_N) = \mathcal{E}_s(S^d, N),$$

then  $\omega_N$  is called a minimal  $s$ -energy configuration, and the points in  $\omega_N$  are called minimal  $s$ -energy points, or simply minimal energy points if the linkage to the parameter  $s$  is well-understood in a certain context. It is obvious that minimal  $s$ -energy configurations exist. Also, if  $\omega_N$  is a minimal  $s$ -energy configuration, and if  $\rho$  is a metric space isometry from  $S^d$  to  $S^d$ , then the image of  $\omega_N$  under  $\rho$ ,  $\rho(\omega_N)$ , is also a minimal  $s$ -energy configuration. Minimal  $(d-1)$ -energy points are often referred to as “Fekete points”; see [5]. The determination of minimal  $s$ -energy configurations and the corresponding minimal  $s$ -energy on  $S^d$  and other manifolds is an important problem that has applications in many subjects including physics, chemistry, computer science, and mathematics. For further background regarding this problem and its applications, we refer readers to the expository papers by Hardin and Saff [13], and by Saff and Kuijlaars [20]. The papers [6], [9], [12], [14], and [18] and the references therein also contain valuable pertinent information.

The determination of the distribution of minimal  $s$ -energy points in the cases  $d \geq 2$  turns out to be rather elusive. It is, however, generally expected that these points are “well-separated” in the sense that there exists a positive constant  $A_{d,s}$ , depending only on  $d$  and  $s$ , such that

$$\min_{i \neq j} |x_i - x_j| \geq A_{d,s} N^{-1/d}. \quad (1.2)$$

Dahlberg [5] proved that Fekete points are well-separated. (Dahlberg [5] actually established the well-separatedness of Fekete points on every compact  $C^{1,\alpha}$ -surface in  $\mathbb{R}^{d+1}$ ). Kuijlaars and Saff [14] proved that minimal  $s$ -energy points are well-separated for the cases  $s > d$ . There have been a series of more quantitative results regarding the case  $d = 2, s = 0$ , which corresponds to the minimal logarithmic energy on  $S^2$ . Rakhmanov, Saff, and Zhou [19] first showed that  $3/5$  is a lower bound for the constant  $A_{2,0}$  in inequality (1.2). Dubickas [8] refined Rakhmanov, Saff, and Zhou’s method, and showed that  $7/4$  is a lower bound for the constant  $A_{2,0}$  in inequality (1.2). Using potential theory and stereographical projection techniques, Dragnev [7] established an appealing lower bound for the minimum separation of the minimal logarithmic energy points on  $S^2$  to be  $2(N-1)^{-1/2}$ .

In this paper, we show that the minimal  $s$ -energy points are well-separated for the cases  $d-1 \leq s < d$ . Note that the case  $s = d-1$  is already covered by the aforementioned Dahlberg’s result. While the parameter  $s$  is restricted in the range  $d-1 \leq s < d$ , the outcome  $s = 0$  can only occur when  $d = 1$ , putting the problem

on the unit circle  $S^1$ . It is shown by Götz [10] that the  $N$ th roots of unity and their rotations are the only minimal  $s$ -energy configurations on  $S^1$ ; see also [16]. Therefore in this paper, we can use the tacit assumption that  $s > 0$ . Our proof entails comparing the Riesz  $s$ -potentials on the slightly larger sphere of radius  $1 + N^{-1/d}$  of two probability measures: the rotationally invariant probability measure on  $S^d$ , and the normalized counting measure on a minimal  $s$ -energy configuration  $\omega_N$ . The Riesz  $s$ -potential of the rotationally invariant probability measure on  $S^d$  can be expressed in closed form in terms of the Gauss hypergeometric functions  ${}_2F_1(a, b; c; z)$ . The crux of our argument is an application of the principle of domination for  $\alpha$ -superharmonic functions; see [15]. This paper is organized as follows. In Section 2, we introduce the necessary notations and terminologies. Also in Section 2, we list a few formulas pertaining to the hypergeometric functions that we will use in our proofs. In Section 3, we state and prove our main result.

## 2 Notation and terminology.

Given a minimal  $s$ -energy configuration  $\omega_N$ , we use  $v_N$  to denote the normalized counting measure on  $\omega_N$ , i.e.,

$$v_N := N^{-1} \sum_{j=1}^N \delta_{x_j},$$

where  $\delta_{x_j}$  denotes the unit point mass at  $x_j$ . The rotationally invariant probability measure on  $S^d$  is denoted by  $\mu$ . For a  $\sigma$ -finite positive Borel measure  $\lambda$  supported on a compact subset  $K$  of  $\mathbb{R}^{d+1}$ , we define its Riesz  $s$ -potential  $U_s^\lambda$  ( $s > 0$ ) by

$$U_s^\lambda(x) := \int_K |x - y|^{-s} d\lambda(y).$$

Note that  $U_s^\lambda$  may take the extended value  $\infty$  on some subsets of  $\mathbb{R}^{d+1}$ . In this paper,  $K$  is either  $S^d$  or  $\omega_N$ . In the next section, we will show that the Riesz  $s$ -potential of  $\mu$  can be expressed in closed form in terms of the Gauss hypergeometric functions  ${}_2F_1(a, b; c; z)$  defined by

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where  $(a)_n$  is the Pochhammer symbol defined by

$$(a)_n := \begin{cases} 1, & \text{if } n = 0, \\ a(a+1) \cdots (a+n-1), & \text{if } n \geq 1. \end{cases}$$

There are several sources that provide essential properties of the hypergeometric functions  ${}_2F_1(a, b; c; z)$  such as Abramowitz et al [1], and Andrews et al [2]. We will be using a few basic formulas pertaining to the hypergeometric functions  ${}_2F_1(a, b; c; z)$ , which can all be found in [1]. We quote them here for easy reference. Under the conditions  $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ , the following formula holds true:

$$\int_0^1 (1 - zu)^{-a} u^{b-1} (1 - u)^{c-b-1} du = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z). \quad (2.1)$$

The above integral represents an analytic function in the  $z$ -plane cut along the real axis from 1 to  $\infty$ . Formula (2.1) is often called Euler's integral representation for the hypergeometric function  ${}_2F_1$ . If  $\operatorname{Re}(c - a - b) < 0$ , then

$$\lim_{z \rightarrow 1^-} \frac{{}_2F_1(a, b; c; z)}{(1 - z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}. \quad (2.2)$$

If  $\operatorname{Re}(c - a - b) > 0$ , then

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \quad (2.3)$$

The following derivative formula can be easily proved by term-by-term differentiation in a suitable domain:

$$\frac{d}{dz} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a + 1, b + 1; c + 1; z). \quad (2.4)$$

### 3 Main Result and Proofs.

On various occasions, we use  $C_{d,s}$  to denote some unspecified positive constants, depending only on  $d$  and  $s$ . The exact values of  $C_{d,s}$  may be different from proof to proof. In the same proof, however, for clarity we use different notations for different constants, namely  $C'_{d,s}$ , and  $C''_{d,s}$  if necessary. Although in the current paper, we do not strive to estimate these constants, closed forms for them can be obtained with some devoted calculations. We will be using the notations  $\sum_{i \neq j}$ , and  $\sum_{j:j \neq i}$ . The former denotes a “double sum”, excluding only those terms with  $i = j$ . The latter denotes a single sum in which  $i$  is fixed, and the summation is done only on  $j$ .

The following lemma is well-known; see e.g. [14].

**Lemma 1.** *For  $d - 1 \leq s < d$ , there exists a positive constant  $C_{d,s}$  independent of  $N$ , such that for any  $N$  distinct points  $x_1, \dots, x_N$  on  $S^d$ , we have*

$$\sum_{i \neq j} |x_i - x_j|^{-s} > \gamma_{d,s} N^2 - C_{d,s} N^{1+s/d}, \quad (3.1)$$

where

$$\gamma_{d,s} := \int_{S^d} \int_{S^d} |x - y|^{-s} d\mu(x) d\mu(y) = \frac{\Gamma((d+1)/2) \Gamma(d-s)}{\Gamma((d-s+1)/2) \Gamma(d-s/2)}.$$

By inspecting the pertinent proof in [14], we find that a quantitative estimate of the constant  $C_{d,s}$  in the above lemma is possible. The determination of asymptotically sharp values for these constants has received much attention in the literature; see [3], [14], [19], and references therein.

**Lemma 2.** *For  $0 < s < d$ , there exists a positive constant  $C_{d,s}$  independent of  $N$ , such that*

$$U_s^{v_N}(x) \geq \gamma_{d,s} - C_{d,s} N^{-1+s/d}, \quad |x| = 1. \quad (3.2)$$

**Proof:** Since  $v_N$  is the normalized counting measure of a minimal  $s$ -energy configuration  $\omega_N = \{x_1, \dots, x_N\}$ , we have for each fixed  $i$ ,  $1 \leq i \leq N$ ,

$$\sum_{j:j \neq i} |x - x_j|^{-s} \geq \sum_{j:j \neq i} |x_i - x_j|^{-s}, \quad x \in S^d.$$

Summing over  $i$  and using Lemma 1, we get

$$(N-1) \sum_{j=1}^N |x - x_j|^{-s} \geq \sum_{i \neq j} |x_i - x_j|^{-s} \geq \gamma_{d,s} N^2 - C_{d,s} N^{1+s/d}. \quad (3.3)$$

Dividing by  $N(N-1)$  we get the desired estimate.  $\blacksquare$

**Lemma 3.** *For each fixed  $s > 0$ , the potential  $U_s^\mu$  of the measure  $\mu$  is a radial function, and has the explicit expression in terms of the hypergeometric function:*

$$U_s^\mu(x) = (R+1)^{-s} {}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; \frac{4R}{(R+1)^2}\right), \quad |x| = R \neq 1.$$

**Proof:** For  $x \in \mathbb{R}^{d+1}$ ,  $|x| \neq 1$ , we have

$$U_s^\mu(x) = \int_{S^d} |x - y|^{-s} d\mu(y).$$

Let  $x, y \in \mathbb{R}^{d+1}$  with  $|x| = R$  and  $|y| = 1$ . Denote the angle between the two vectors  $x$  and  $y$  by  $\theta$ . Then  $\cos \theta = \langle \frac{x}{R}, y \rangle$ . By the law of cosine,  $|x - y|^2 = R^2 + 1 - 2R\langle \frac{x}{R}, y \rangle$ . Thus by using the Funk-Hecke formula; see Müller [17], we have

$$\begin{aligned} \int_{S^d} |x - y|^{-s} d\mu(y) &= \int_{S^d} (R^2 + 1 - 2R\langle \frac{x}{R}, y \rangle)^{-s/2} d\mu(y) \\ &= \frac{\nu_{d-1}}{\nu_d} \int_{-1}^1 (R^2 + 1 - 2Rt)^{-s/2} (1-t^2)^{(d-2)/2} dt, \end{aligned}$$

where  $\nu_d$  denotes the surface area of  $S^d$ . Using the substitution  $2u = t + 1$  and Euler's integral representation of the hypergeometric function  ${}_2F_1$ , we have

$$\begin{aligned} U_s^\mu(x) &= 2^{d-1}(R+1)^{-s} \frac{\nu_{d-1}}{\nu_d} \int_0^1 \left(1 - \frac{4R}{(R+1)^2}u\right)^{-s/2} u^{(d-2)/2} (1-u)^{(d-2)/2} du \\ &= 2^{d-1}(R+1)^{-s} \frac{\nu_{d-1}}{\nu_d} \frac{\Gamma^2(\frac{d}{2})}{\Gamma(d)} {}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; \frac{4R}{(R+1)^2}\right). \end{aligned}$$

To simplify, we use the formula (see [17])

$$\nu_d = \frac{2\pi^{(d+1)/2}}{\Gamma(\frac{d+1}{2})},$$

and then the formula (see [1])

$$\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2} \Gamma(z) \Gamma(z+1/2), \quad (3.4)$$

with  $z = d/2$ . We have

$$\begin{aligned} U_s^\mu(x) &= 2^{d-1}(R+1)^{-s} \frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2})} \frac{\Gamma^2(\frac{d}{2})}{\Gamma(d)} {}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; \frac{4R}{(R+1)^2}\right) \\ &= (R+1)^{-s} {}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; \frac{4R}{(R+1)^2}\right). \quad \blacksquare \end{aligned}$$

The following two special cases of Lemma 3 are worth noting. Firstly, when  $0 < s < d$ , the potential  $U_s^\mu(x)$  is well defined for  $|x| = 1$ . In fact, a simple application of the Lebesgue Dominated Convergence Theorem yields

$$\lim_{|x| \rightarrow 1} U_s^\mu(x) = \lim_{|x| \rightarrow 1} \int_{S^d} |x - y|^{-s} d\mu(y) = \gamma_{d,s}.$$

On the other hand, when  $0 < s < d$ , the hypergeometric function  ${}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; z\right)$  is continuous at  $z = 1$ . Using equation (2.3), we have

$$\begin{aligned} \lim_{R \rightarrow 1} (R+1)^{-s} {}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; \frac{4R}{(R+1)^2}\right) &= 2^{-s} {}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; 1\right) \\ &= 2^{-s} \frac{\Gamma(d)\Gamma(\frac{d-s}{2})}{\Gamma(d/2)\Gamma(d-s/2)}. \end{aligned}$$

Thus, we have

$$\gamma_{d,s} = 2^{-s} \frac{\Gamma(d)\Gamma(\frac{d-s}{2})}{\Gamma(d/2)\Gamma(d-s/2)}. \quad (3.5)$$

Due to the importance of the constant  $\gamma_{d,s}$  in this paper, we feel reassured that we are able to verify equation (3.5) directly, and we share the reassurance with the readers. Using equation (3.4) with  $z = d/2$ , we write

$$\Gamma(d) = (2\pi)^{-1/2} 2^{d-1/2} \Gamma(d/2) \Gamma((d+1)/2),$$

we have

$$\begin{aligned}
2^{-s} \frac{\Gamma(d)\Gamma(\frac{d-s}{2})}{\Gamma(d/2)\Gamma(d-s/2)} &= 2^{-s} \frac{(2\pi)^{-1/2} 2^{d-1/2} \Gamma(d/2)\Gamma((d+1)/2)\Gamma(\frac{d-s}{2})}{\Gamma(d/2)\Gamma(d-s/2)} \\
&= \frac{(2\pi)^{-1/2} 2^{d-s-1/2} \Gamma(\frac{d-s}{2})\Gamma(\frac{d-s+1}{2})\Gamma((d+1)/2)}{\Gamma(\frac{d-s+1}{2})\Gamma(d-s/2)} \\
&= \frac{\Gamma(d-s)\Gamma((d+1)/2)}{\Gamma(\frac{d-s+1}{2})\Gamma(d-s/2)}.
\end{aligned}$$

Here in the last step, we have used equation (3.4) again with  $z = (d-s)/2$ . Secondly, when  $d = 2$ , the potential  $U_s^\mu(x)$  has the elementary form:

$$U_s^\mu(x) = \frac{1}{2R} \frac{(1+R)^{2-s} - |R-1|^{2-s}}{2-s}, \quad |x| = R, \quad s \neq 2.$$

**Lemma 4.** *Assume  $d-1 \leq s < d$ . Then there exists a positive constant  $C_{d,s}$ , independent of  $N$ , such that*

$$U_s^\mu(x) > \gamma_{d,s} - C_{d,s} N^{-1+s/d},$$

for all  $x \in \mathbb{R}^{d+1}$  with  $|x| = 1 + N^{-1/d}$ .

**Proof:** We first note that with  $R_{N,d} := 1 + N^{-1/d}$ , we have

$$(R_{N,d} + 1)^{-s} = 2^{-s} \left(1 - \frac{s}{2} N^{-1/d}\right) + o(N^{-1/d}). \quad (3.6)$$

We now estimate  ${}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; \frac{4R_{N,d}}{(R_{N,d}+1)^2}\right)$ . By using the Fundamental Theorem of Calculus and then equation (2.4), we have

$$\begin{aligned}
&{}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; \frac{4R_{N,d}}{(R_{N,d}+1)^2}\right) \\
&= {}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; 1\right) - \left[ {}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; 1\right) - {}_2F_1\left(\frac{s}{2}, \frac{d}{2}; d; \frac{4R_{N,d}}{(R_{N,d}+1)^2}\right) \right] \\
&= \frac{\Gamma(d)\Gamma(\frac{d-s}{2})}{\Gamma(d/2)\Gamma(d-s/2)} - \frac{s}{4} \int_{4R_{N,d}/(R_{N,d}+1)^2}^1 {}_2F_1\left(\frac{s}{2} + 1, \frac{d}{2} + 1; d + 1; z\right) dz.
\end{aligned}$$

We use equation (2.2) to estimate the above integral. For any given  $\epsilon > 0$ , we have, for  $N$  sufficiently large, that

$$\left| \frac{{}_2F_1\left(\frac{s}{2} + 1, \frac{d}{2} + 1; d + 1; z\right)}{(1-z)^{(d-s-2)/2}} - \beta_{d,s} \right| < \epsilon, \quad z \in [4R_{N,d}/(R_{N,d}+1)^2, 1],$$

where

$$\beta_{d,s} := \frac{\Gamma(d+1)\Gamma((d-s)/2+1)}{\Gamma(d/2+1)\Gamma(s/2+1)}.$$

This implies

$$\begin{aligned}
& \int_{4R_{N,d}/(R_{N,d}+1)^2}^1 {}_2F_1\left(\frac{s}{2}+1, \frac{d}{2}+1; d+1; z\right) dz \\
& \leq (\beta_{d,s} + \epsilon) \int_{4R_{N,d}/(R_{N,d}+1)^2}^1 (1-z)^{(d-s-2)/2} dz \\
& = (\beta_{d,s} + \epsilon) \frac{2}{d-s} \left(\frac{R_{N,d}-1}{R_{N,d}+1}\right)^{d-s} \\
& \leq \frac{2^{s-d+1}}{d-s} (\beta_{d,s} + \epsilon) N^{-1+s/d}.
\end{aligned} \tag{3.7}$$

Combining Lemma 3 and the above two estimates (3.6) and (3.7), and noting that  $1/d \geq 1 - s/d$ , we have for  $x$  with  $|x| = R_{N,d}$ ,

$$U_s^\mu(x) = 2^{-s} \left(1 - \frac{s}{2} N^{-1/d}\right) \left( \frac{\Gamma(d)\Gamma(\frac{d-s}{2})}{\Gamma(d/2)\Gamma(d-s/2)} - C_{d,s} N^{-1+s/d} \right) + o(N^{-1+s/d}),$$

which gives the desired result of Lemma 4. ■

In the discussion that follows, we will need the notion of “ $\alpha$ -superharmonic functions” and the principal of domination of  $\alpha$ -superharmonic functions. These topics can be found in Landkof [15]. The definition of  $\alpha$ -superharmonic functions is technical. Upon checking the pertinent material in Landkof [15], one finds the relation  $d+1-\alpha=s$  between the parameter  $\alpha$  used in Landkof [15] and the parameter  $s$  we use here. Thus the requirement  $d-1 \leq s < d$  translates into  $1 < \alpha \leq 2$  in Landkof [15]. Furthermore, what is meant in Landkof [15] by an  $\alpha$ -superharmonic function is in fact a  $(d+1-s)$ -superharmonic function here in our context. We choose not to use the phrase  $(d+1-s)$ -superharmonic function because we feel that notion  $\alpha$ -superharmonic function has been coined in the mathematical literature. In Chapter 1, Section 5 of [15], it is proved that for a  $\sigma$ -finite Borel measure  $\lambda$  supported on a compact subset of  $\mathbb{R}^{d+1}$ , its potential  $U_s^\lambda$  ( $0 \leq s < d$ ) is an  $\alpha$ -superharmonic function. Furthermore, Theorem 1.29 in [15], aside from some changes in notation, states the following result:

**Theorem 5.** *Suppose  $\lambda$  is a  $\sigma$ -finite positive Borel measure supported on a compact subset of  $\mathbb{R}^{d+1}$  whose potential  $U_s^\lambda$  is finite  $\lambda$ -almost everywhere, and that  $f(x)$  is an  $\alpha$ -superharmonic function. If the inequality*

$$U_s^\lambda(x) \leq f(x)$$

*holds  $\lambda$ -almost everywhere, then it holds everywhere in  $\mathbb{R}^{d+1}$ .*

Theorem 5 is often called the “principal of domination” for  $\alpha$ -superharmonic functions.

**Lemma 6.** Assume  $d - 1 \leq s < d$ . Then there exists a constant  $C_{d,s}$ , independent of  $N$ , such that for all  $x \in \mathbb{R}^{d+1}$  with  $|x| = 1 + N^{-1/d}$ ,

$$U_s^{v_N}(x) \geq \gamma_{d,s} - C_{d,s}N^{-1+s/d}. \quad (3.8)$$

**Proof:** By Lemma 2, we have

$$U_s^{v_N}(x) \geq \gamma_{d,s} - C_{d,s}N^{-1+s/d}, \quad |x| = 1.$$

Note that  $U_s^\mu(x) = \gamma_{d,s}$ ,  $x \in S^d$ , so we can rewrite the above inequality as

$$U_s^{v_N}(x) \geq U_s^\mu(x)(1 - C'_{d,s}N^{-1+s/d}), \quad |x| = 1.$$

Both measures  $\mu$  and  $v_N$  are supported on  $S^d$ . Since  $U_s^{v_N}$  is an  $\alpha$ -superharmonic function, we can therefore use Theorem 5 to obtain for  $N$  sufficiently large,

$$U_s^{v_N}(x) \geq U_s^\mu(x)(1 - C'_{d,s}N^{-1+s/d}), \quad x \in \mathbb{R}^{d+1}.$$

We then use Lemma 4 to get for  $|x| = 1 + N^{-1/d}$ ,

$$U_s^{v_N}(x) \geq (\gamma_{d,s} - C''_{d,s}N^{-1+s/d})(1 - C'_{d,s}N^{-1+s/d}),$$

which yields the desired estimate. ■

**Lemma 7.** For every fixed  $i$ ,  $(1 \leq i \leq N)$ , we have, for  $0 < s < d$ ,

$$N^{-1} \sum_{j:j \neq i} |x_i - x_j|^{-s} \leq \gamma_{d,s}.$$

**Proof:** For every fixed  $i$ ,  $(1 \leq i \leq N)$ , since the function  $x \mapsto \sum_{j:j \neq i} |x - x_j|^{-s}$  reaches its minimum at  $x_i$  on  $S^d$ , we have

$$N^{-1} \sum_{j:j \neq i} |x_i - x_j|^{-s} \leq N^{-1} \sum_{j:j \neq i} |x - x_j|^{-s}, \quad x \in S^d.$$

Integrating both sides of the above inequality on  $S^d$  against the probability measure  $\mu(x)$  yields:

$$N^{-1} \sum_{j:j \neq i} |x_i - x_j|^{-s} \leq N^{-1} \int_{S^d} \sum_{j:j \neq i} |x - x_j|^{-s} d\mu(x) = \frac{N-1}{N} \gamma_{d,s},$$

and therefore the desired inequality follows. ■

**Theorem 8.** Assume  $d - 1 \leq s < d$ . The minimal  $s$ -energy points are well-separated, i.e., there exists a constant  $A_{d,s} > 0$ , independent of  $N$ , such that

$$\min_{i \neq j} |x_i - x_j| \geq A_{d,s}N^{-1/d}.$$

**Proof:** Let  $\omega_N$  be a minimal  $s$ -energy configuration, and let  $x_{i_0}, x_{j_0}$  be two points in  $\omega_N$  such that

$$|x_{i_0} - x_{j_0}| = \min_{i \neq j} |x_i - x_j|.$$

Using Lemma 7 we have

$$\begin{aligned} \gamma_{d,s} - N^{-1}|x_{i_0} - x_{j_0}|^{-s} &\geq N^{-1} \sum_{j:j \neq i_0} |x_{i_0} - x_j|^{-s} - N^{-1}|x_{i_0} - x_{j_0}|^{-s} \\ &= N^{-1} \sum_{j:j \neq i_0, j_0} |x_{i_0} - x_j|^{-s}. \end{aligned} \quad (3.9)$$

Take  $x := (1 + N^{-1/d})x_{i_0}$ . Then  $|x_{i_0} - x_j| < |x - x_j|$  for every  $j$ , and so by (3.9) and Lemma 6 we have

$$\begin{aligned} \gamma_{d,s} - N^{-1}|x_{i_0} - x_{j_0}|^{-s} &\geq N^{-1} \sum_{j:j \neq i_0, j_0} |x - x_j|^{-s} \\ &= U_s^{v_N}(x) - N^{-1}|x - x_{i_0}|^{-s} - N^{-1}|x - x_{j_0}|^{-s} \\ &\geq \gamma_{d,s} - C_{d,s}N^{-1+s/d} - N^{-1}|x - x_{i_0}|^{-s} - N^{-1}|x - x_{j_0}|^{-s}. \end{aligned}$$

Since  $|x - x_{i_0}| = N^{-1/d}$  and  $|x - x_{j_0}| > N^{-1/d}$ , it follows that

$$\gamma_{d,s} - N^{-1}|x_{i_0} - x_{j_0}|^{-s} \geq \gamma_{d,s} - (C_{d,s} + 2)N^{-1+s/d},$$

which implies that for some constant  $A_{d,s}$ ,

$$|x_{i_0} - x_{j_0}| \geq A_{d,s}N^{-1/d}. \quad \blacksquare$$

If one follows the trail of the constants throughout the proofs, then one is able to quantitatively estimate the value of the constant  $A_{d,s}$  in Theorem 8. However, such a process leads to the piling-up of many Gamma function values, among others. There seems to be no obvious way to simplify them. Just for curiosity, we numerically estimated the constant  $A_{d,s}$  in Theorem 8 for the case  $d = 2$ , and  $s = 1$ . Our numerical result yields  $A_{2,1} \geq 0.8709$ , putting our estimate of the minimum separation of the corresponding minimal energy points on  $S^2$  at  $0.8709/\sqrt{N}$ . The result of Habicht and Van der Waerden [11] for best packing asserts that the maximum diameter of  $N$  non-overlapping congruent circles on  $S^2$  is asymptotically

$$\left(\frac{8\pi}{\sqrt{3}}\right)^{1/2} \frac{1}{\sqrt{N}} \approx 3.809 \frac{1}{\sqrt{N}}.$$

For more information on best packing on  $S^2$ , we also refer readers to [4], [20], and the references therein.

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